

# Generalized Broughton polynomials and characteristic varieties

NGUYEN TAT THANG  
 Institute of Mathematics  
 18 Hoang Quoc Viet road  
 Cau Giay District, 10307 Hanoi, Vietnam  
 ntthang@math.ac.vn

## Abstract

We introduce a family of generalized Broughton polynomials and compute the characteristic varieties of complement of a curve arrangement defined by fibers of some generalized Broughton polynomials.

## 1 Introduction

In [Br] Broughton considered the polynomial

$$f(x, y) = x(xy - 1).$$

The associated function  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  has no critical value, but the fiber  $f^{-1}(0)$  is not diffeomorphic to the generic one. This is explained by the existence of the so-called "critical value at infinity", see [HL], [Br], [D1].

In the paper [Z] Zahid introduced a family of polynomials:

$$f_{p,q}(x, y) = x^p[xy(x + 2) \cdots (x + q) - 1],$$

which are called *generalized Broughton polynomials*, where  $p \geq 1$  and  $q \geq 1$  are integer number, with the convention

$$f_{p,1} = x^p(xy - 1).$$

By computing the characteristic variety  $\mathcal{V}_1(M)$ , where

$$M = \mathbb{C}^2 \setminus (C_0 \cup C_1)$$

---

*Key words and phrases.* Broughton polynomial, characteristic varieties, translated component, connected generic fiber.

2010 *Mathematics subject classification.* Primary 14C21, 14H50; Secondary 14H20, 32S22.

is a complement of a curve arrangement defined by a component of the 0-fiber:

$$C_0 = \begin{cases} \{xy(x+2) \cdots (x+q) - 1 = 0\} & \text{if } q > 1 \\ \{xy - 1 = 0\} & \text{if } q = 1 \end{cases}$$

and the generic fiber of  $f_{p,q}$ :

$$C_1 = \{f_{p,q}(x, y) = 1\},$$

the author obtained examples of characteristic varieties with an arbitrary number of translated components for complements of affine curve arrangements consisting of just two rational curves, see [Z].

The aim of this paper is to generalize the Zahid's work in [Z]. More precisely, we introduce a family of *generalized Broughton polynomials*, which generalizes the Zahid's one. Namely

$$F(x, y) := p(x)(yq(x) - 1)$$

where  $p(x), q(x) \in \mathbb{C}[x]$ .

Put  $f(x, y) := F(x, y) - 1$  and  $g(x, y) := yq(x) - 1$ . We denote by  $M$  the complement

$$M = \mathbb{C}^2 \setminus \{f(x, y) = 0, g(x, y) = 0\}.$$

The main result in this note shows how to compute the characteristic variety  $\mathcal{V}_1(M)$ , for all polynomials  $p(x), q(x)$  such that they have at least one common root and  $p(x) + 1, q(x)$  have not any common root.

In Section 2 we recall the definition and the basic properties of the characteristic and resonance varieties. In Section 3 we compute the characteristic variety  $\mathcal{V}_1(M)$ . In particular, we obtain examples of characteristic varieties with an arbitrary number of translated components (Theorem 3.6). This is an extension for Theorem 4.1 in [Z].

## 2 Characteristic and Resonance varieties

Let  $M$  be a smooth, irreducible, quasi-projective complex variety. The characteristic variety of  $M$  is defined by

$$\mathbb{T}(M) := \text{Hom}(H_1(M), \mathbb{C}^*).$$

This is an algebraic group whose identity irreducible component  $\mathbb{T}(M)_1$  is an algebraic torus  $(\mathbb{C}^*)^{b_1(M)}$ . Consider the exponential mapping

$$\exp : H^1(M, \mathbb{C}) \rightarrow H^1(M, \mathbb{C}^*) = \mathbb{T}(M) \tag{1}$$

induced by the usual exponential function  $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ . Clearly

$$\exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)_1.$$

The *characteristic varieties* of  $M$  are the jumping loci for the first cohomology of  $M$ , with coefficients in rank one local systems:

$$\mathcal{V}_k^i(M) = \{\rho \in \mathbb{T}(M) : \dim H^i(M, \mathcal{L}_\rho) \geq k\}.$$

When  $i = 1$ , we use the simpler notation  $\mathcal{V}_k(M) = \mathcal{V}_k^1(M)$ .

Fundamental results on the structure of the cohomology support loci for local systems on quasi-projective algebraic varieties were obtained by Beauville [Be], Green and Lazarsfeld [GL], Simpson [S] (for the proper case), and Arapura [A] (for the quasi-projective case and first characteristic varieties  $\mathcal{V}_1(M)$ ).

**Theorem 2.1.** *The strictly positive dimensional irreducible components of the first characteristic variety  $\mathcal{V}_1(M)$  are translated subtori in  $\mathbb{T}(M)$  by elements of finite order. When  $M$  is proper, all the components of  $\mathcal{V}_k^i(M)$  are translated subtori in  $\mathbb{T}(M)$  by elements of finite order.*

The strictly positive dimensional irreducible components of the first characteristic variety  $\mathcal{V}_1(M)$  are described as follows.

**Theorem 2.2.** *([Be], [A]) Let  $W$  be a  $d$ -dimensional irreducible component of  $\mathcal{V}_1(M)$ ,  $d > 0$ . Then, there is a regular morphism  $f : M \rightarrow S$  onto a smooth curve  $S$  with  $b_1(S) = d$  such that the generic fiber  $F$  of  $f$  is connected, and a torsion character  $\rho \in \mathbb{T}(M)$  such that the composition*

$$\pi_1(F) \rightarrow \pi_1(M) \rightarrow \mathbb{C}^*$$

*is trivial and  $W = \rho \cdot f^*(\mathbb{T}(S))$ .*

**Remark 2.3.** When  $M$  is a hypersurface complement in  $\mathbb{P}^n$ , the curve  $S$  in Theorem 2.2 above is obtained from  $\mathbb{C}$  by deleting  $d$  points, see [D4], Theorem 1.11.

If we fix a regular mapping  $f : M \rightarrow S$  as above, the number of irreducible components  $W = \rho \cdot f^*(\mathbb{T}(S))$  obtained by varying the torsion character  $\rho$  is given by the following.

**Theorem 2.4.** *([D3]) For a given regular map  $f : M \rightarrow S$  as above, the associated irreducible components  $W = \rho \cdot f^*(\mathbb{T}(S))$  are parametrized by the Pontrjagin dual  $\hat{T}(f) = \text{Hom}(T(f); \mathbb{C}^*)$  of the finite abelian group*

$$T(f) = \frac{\ker\{f^* : H_1(M) \rightarrow H_1(S)\}}{\text{im}\{i^* : H_1(F) \rightarrow H_1(M)\}}$$

*if  $\chi(S) < 0$  and by the non-trivial elements of this Pontrjagin dual  $\hat{T}(f)$  if  $\chi(S) = 0$ .*

The group  $T(f)$  is determined as follows.

**Theorem 2.5.** *([D3]) Let  $S$  is a non-proper smooth curve and  $f : M \rightarrow S$  be a regular function. Then the group  $T(f)$  is computed by the following*

$$T(f) = \oplus_{c \in C(f)} \mathbb{Z}/m_c \mathbb{Z},$$

*where  $m_c$  is the multiplicity of the divisor  $f^{-1}(c)$  and  $C(f)$  is the set of bifurcation values of  $f$ .*

The (first) *resonance varieties* of  $M$  are the jumping loci for the first cohomology of the complex  $H^*(H^*(M, \mathbb{C}), \alpha \wedge)$ , namely

$$\mathcal{R}_k(M) = \{\alpha \in H^1(M, \mathbb{C}) : \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge) \geq k\}.$$

The relation between the resonance and characteristic varieties can be summarized as follows, see [D5].

**Theorem 2.6.** *Assume that  $M$  is any hypersurface complement in  $\mathbb{P}^n$ . Then the irreducible components  $E$  of the resonance variety  $\mathcal{R}_1(M)$  are linear subspaces in  $H^1(M, \mathbb{C})$  and the exponential mapping (1) sends these irreducible components  $E$  onto the irreducible components  $W$  of  $\mathcal{R}_1(M)$  with  $1 \in W$ .*

### 3 The Characteristic varieties $\mathcal{V}_1(M)$

Consider from now on the complement  $M = \mathbb{C}^2 \setminus C$ , where  $C = C_0 \cup C_1$ ,  $C_0 = \{g(x, y) = 0\}$  and  $C_1 = \{f(x, y) = 0\}$ .

By the same argument as in Section 3 in [Z] we can prove the following.

**Theorem 3.1.** *The integral (co)homology of the surface  $M$  is torsion free and*

$$b_1(M) = 2, b_2(M) = s + t,$$

where  $s$  and  $t$  are the numbers of roots of  $q(x)$  and  $p(x)q(x)$ , respectively. Moreover, the cup-product

$$\cup : H^1(M) \times H^1(M) \rightarrow H^2(M)$$

is non-trivial.

Using the definition of the resonance varieties we get the following.

**Corollary 3.2.** *The resonance varieties of  $M$  are trivial, i.e.  $\mathcal{R}_k(M) = 0$  for any  $k > 0$ .*

Since the resonance varieties are trivial, and  $M$  is a hypersurface complement, it follows from Theorem 2.6 that the characteristic varieties  $\mathcal{V}_1(M)$  can contain only isolated points and 1-dimensional translated components. In this section we determine the latter ones.

In view of Theorem 2.2 and Remark 2.3, any such component comes from a mapping  $h : M \rightarrow \mathbb{C}^*$ . If we regard  $h$  as a regular function on the affine variety  $M$ , it follows that  $h$  should have the form

$$h = \frac{P(x, y)}{f^m g^n}$$

for some polynomial  $P$  and some positive integers  $m, n$ . If  $P$  is not in the multiplicative system spanned by  $f$  and  $g$ , then  $P$  vanishes at some point of  $M$  and this is a contradiction. It follows that we may assume that

$$h = f^m g^n$$

for some (positive or negative) integers  $m, n$ . Now, we are looking for all such mappings such that they have multiple fibers and connected generic fiber.

**Lemma 3.3.** *For all integer numbers  $m > 1, n > 1$  and  $c \in \mathbb{C} \setminus \{0\}$ , then the generic fiber of the polynomial  $f^m(x, y) + cg^n(x, y)$  is connected.*

We need the following fact.

**Lemma 3.4.** *([D3]) For any polynomial map  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  the followings are equivalent:*

- (1) *The generic fiber of  $P$  is connected;*
- (2) *There do not exist polynomials  $H : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C}^n \rightarrow \mathbb{C}$  such that  $\deg(H) > 1$  and  $P = H(Q)$ .*

*Proof of Lemma 3.3.* Let  $\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be given by  $\Phi(x, y) = (x, g(x, y))$ . We have

$$f^m(x, y) + cg^n(x, y) = h \circ \Phi,$$

where  $h(u, v) := (p(u)v - 1)^m + cv^n$ .

It is easy to see that the restriction of  $\Phi$  on  $\mathbb{C}^2 \setminus A$  is a homeomorphism, where  $A = \{(a, y) : q(a) = 0, y \in \mathbb{C}\}$ . Then, the generic fiber of  $f^m(x, y) + cg^n(x, y)$  is connected if and only if the generic fiber of  $h(u, v)$  is connected.

Now, we assume by contradiction that the generic fiber of  $h(u, v)$  is not connected. According to Lemma 3.4, there are polynomials  $H : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C}^2 \rightarrow \mathbb{C}$  such that  $\deg(H) > 1$  and

$$(p(u)v - 1)^m + cv^n = H(Q(u, v)).$$

We consider the singular locus of the polynomials in the above equality. Since  $\deg(H) > 1$  then the singular locus of  $H(Q(u, v))$  has dimension at least one. In particular, there are infinitely many points. However, singular points of  $h(u, v)$  are roots of the following systems.

$$\begin{cases} p'(u) = 0, \\ mp(u)(p(u)v - 1)^{m-1} + cnv^{n-1} = 0 \end{cases}$$

or

$$\begin{cases} p(u) = 0. \\ v = 0 \end{cases}$$

It is easy to see that the above systems have only finitely many points. Contradiction.  $\square$

**Lemma 3.5.** *Assume that the map  $h = f^m g^n : M \rightarrow \mathbb{C}^*$  has connected generic fiber and a multiple fiber. Then  $n = 0$  and  $m = \pm 1$ .*

*Proof.* If  $n = 0$  then  $m = \pm 1$ , because  $h$  has connected generic fiber. Similarly, if  $m = 0$  then  $n = \pm 1$ . However, since  $\deg(f) > \deg(g)$ , it is easy to show that the function  $g : \mathbb{C}^2 \setminus \{fg = 0\} \rightarrow \mathbb{C}^*$  has not any multiple fiber.

Now, we assume that  $mn \neq 0$ . Since  $M = \mathbb{C}^2 \setminus \{fg = 0\}$  and  $f, g$  are two irreducible polynomial then the map  $h : M \rightarrow \mathbb{C}^*$  has multiple fiber if and only if, there exist  $c \in \mathbb{C}^*, h_1 \in \mathbb{C}[x, y], h_1 \nmid f, h_1 \nmid g$  and integer numbers  $s, l, k, |s| > 1$ , such that

$$f^m g^n = c + h_1^s f^l g^k. \quad (2)$$

Since  $f, g, h_1$  are pairwise relatively prime then  $ml \geq 0$  and  $nk \geq 0$ . There are four cases.

a)  $m, l, n, k \geq 0$ : This implies that  $l = k = 0$  and the generic fiber of  $h$  has at least  $|s| > 1$  connected components. Contradiction.

b)  $m, l, n, k \leq 0$ : By dividing two sides of the equality (2) by the lowest powers of  $f$  and  $g$ , one can prove that  $m = l$  and  $n = k$ . It means

$$(f^m g^n)^{-1} = \frac{1}{c}(1 - h_1^s).$$

So the generic fiber of  $f^m g^n$  is not connected.

c)  $m, l \geq 0$  and  $n, k \leq 0$ : Similarly, we get  $l = 0$  and  $n = k$ . Hence  $f^m = cg^{-n} + h_1^s$ . But, this implies that the generic fiber of the polynomial  $f^m - cg^{-n}$  is not connected, contradicts to Lemma 3.3.

d)  $m, l \leq 0$  and  $n, k \geq 0$ : By the same argument, we also obtain the contradiction.  $\square$

The main result in this paper is the following.

**Theorem 3.6.** *Let  $p(x)$  and  $q(x) \in \mathbb{C}[x]$  be two polynomials such that they have at least one common root and  $p(x) + 1, q(x)$  have no common root. Then, if there exist an integer number  $s > 1$  and a polynomial  $p_1 \in \mathbb{C}[x]$  such that*

$$p(x) = p_1(x)^s,$$

*the strictly positive dimensional components of  $\mathcal{V}_1(M)$  are the translated 1-dimensional sub-tori*

$$W_j = \epsilon_j \times \mathbb{C}^*,$$

*where  $d$  is the maximum of the exponent  $s$  above and  $\epsilon_j = \exp(2\pi i j/d)$  for  $j = 1, 2, \dots, d-1$ . Moreover, for a local system  $\mathcal{L} \in W_j$  one has  $\dim H^1(M, \mathcal{L}) \geq 1$  and equality holds with finitely many exceptions.*

*Otherwise, there do not exist strictly positive dimensional components of  $\mathcal{V}_1(M)$ .*

*Proof.* According to Theorem 2.2 and Remark 2.3, any translated positive dimensional component of  $\mathcal{V}_1(M)$  comes from a map  $h : M \rightarrow \mathbb{C}^*$  which has connected generic fiber.

According to Lemma 3.5, the only morphisms associated to strictly positive dimensional components of  $\mathcal{V}_1(M)$  are  $f : M \rightarrow \mathbb{C}^*$  and  $f^{-1} : M \rightarrow$

$\mathbb{C}^*, z \mapsto f(z)^{-1}$ , but they give the same associated component of  $\mathcal{V}_1(M)$ . Thus all translated positive dimensional components of  $\mathcal{V}_1(M)$  are associated to the map  $f : M \rightarrow \mathbb{C}^*$ .

On the other hand, it is easy to see that the only possibly multiple fiber of  $f$  is  $f^{-1}(-1)$ . Hence, according to Theorem 2.5, if  $p(x)$  is not a power of a polynomial then  $T(f) = 0$  and there does not exist strictly positive dimensional components of  $\mathcal{V}_1(M)$ ; unless  $T(f) = \mathbb{Z}/d\mathbb{Z}$ , where

$$d = \max\{s \in \mathbb{N} : p(x) = p_1(x)^s, p_1 \in \mathbb{C}[x]\}.$$

We now consider the later case. It is deduced from Theorem 2.6 that there are exactly  $d - 1$  associated 1-dimensional translated components. If we identify  $\mathbb{T}(M) = \mathbb{C}^*$  by associating to a local system  $\mathcal{L} \in \mathbb{T}(M)$  the two monodromies  $(\lambda_0, \lambda_1)$  about the curves  $C_0$  and  $C_1$ , and in a similar way  $\mathbb{T}(\mathbb{C}^*) = \mathbb{C}^*$ , then the induced morphism

$$f^* : \mathbb{T}(\mathbb{C}^*) \rightarrow \mathbb{T}(M)$$

is just  $\lambda \mapsto (1, \lambda)$ .

With these identifications, the above  $d - 1$  associated 1-dimensional translated components of  $\mathcal{V}_1(M)$  are given by  $W_j = \epsilon_j \times \mathbb{C}^*$ , where  $\epsilon_j = \exp(2\pi i j/d)$  for  $j = 1, 2, \dots, d - 1$ .

The inequality on dimension of cohomology group of  $M$  is the direct consequence of Corollary 5.9 in [D3].  $\square$

## References

- [A] Arapura, D., *Geometry of cohomology support loci for local systems. I*, J. Algebraic Geom., 6 (1997), no. 3, 563-597.
- [Be] Beauville, A., *Annulation du  $H^1$  pour les fibres en droites plats*, in: Complex algebraic varieties (Bayreuth, 1990), 1-15, Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992.
- [Br] Broughton, S. A., *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math., 92 (1988), no. 2, 217- 241.
- [D1] Dimca, A., *Singularities and Topology of Hypersurfaces*, Universitext, Springer, 1992.
- [D2] Dimca, A., *Sheaves in Topology*, Universitext, Springer-Verlag, 2004.
- [D3] Dimca, A., *Characteristic varieties and constructible sheaves*, Rend. Lincei Mat. Appl., 18 (2007), no. 4, 365-389.
- [D4] Dimca, A., *On the irreducible components of characteristic varieties*, An. Stiin. Univ. "Ovidius" Constanta Ser. Mat. 15 (2007), no. 1, 6773.
- [D5] Dimca, A., Papadima, S. and Suciu, A., *Topology and geometry of cohomology jump loci*, Duke Math. J., 148 (2009), no. 3, 405-457.

- [GL] Green, M. and Lazarsfeld, R., *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc., 4 (1991), no. 1, 87- 103.
- [HL] Hà, H. V. and Lê, D. T., *Sur la topologie des polynomes complexes*, Acta Math. Vietnamica, 9 (1984), 21- 32.
- [S] Simpson, C., *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. Ecole Norm. Sup. 26(1993), no. 3, 361401.
- [Z] Zahid, R., *Broughton polynomials and characteristic varieties*, Studia Sci. Math. Hungarica, 47 (2010), no.2, 214-222.